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AVOIDENCE STRATEGIES FOR FRACTIONAL ORDER SYSTEMS WITH CAPUTO DERIVATIVE

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Abstract: A control strategy is derived for fractional-order dynamic systems with Caputo derivative to guarantee collision-free trajectories for two agents. To guarantee that one agent keeps the state of the system out of a given set regardless of the other agent's actions a Lyapunov-based approach is adopted. As a special case showing that the given approach to choosing proposed strategy is constructive for a fractional-order system with the Caputo derivative, a linear system as an example is discussed. Obtained results extend to the fractional order case the avoidance problem Leitman's and Skowronski's approach.

Key words: Avoidence; fractional order systems, Caputo derivative, Lyapunov stability

1. INTRODUCTION

Avoidance system, in a colloquial sense means "a safety system designed to warn, alert, or assist drivers to avoid imminent collisions and reduce the risk of incidents". In fact the subject of collision avoidance is much broader than just pre-crash systems. One only has to look at the game as a conflict situation in which the parties to the conflict choose the strategy suits them, see for example [16]. So, this problem can be met not only in problems of traffic and transport [19,24], navigation [15] but also in communication networks, economic, control [18,25,28], social sciences [6], and others. Generally the problem for the case of two agents is the following: There are two agents. How to determine the strategy of one of them in such way that for given initial state no solution of one system intersects the avoidance zone no matter what a strategy of the second is?

Several approaches to avoidance control have been considered. The Lyapunov based approach as the first was proposed in [14] and next generalized to the system with an arbitrary time domain [22]. Based on [14] conditions for collision avoidance between two agents were given in a non-cooperative case in [8]. The cooperative control law using the concept of Lyapunov function for multi-agent system were designed in [27] and next extended to multi-agent system with bounded input disturbances in [27,29].

Taking into account accelerating development of both fractional differential equations and their applications, the other extension of approach to collision avoidance is important. This derives from the fact that fractional-order equations are more adequate for modeling physical processes than differential equations with an integer order and provides} some explanation of discontinuity and singularity formations in nature, see [9,11]. One can find many applications of fractional calculus and control in viscoelasticity, electrochemistry, electromagnetism, ecnophysics, and others, see for example [1,12,13,23,26]. It cannot be ignore that many modeled systems contain non-local dynamics, which can be better described using integro-differential operators with a fractional order, [9,10,20,21]. Attention should be also paid on the fact when a real phenomenon is mathematically modelled, not all variables are precisely known. This implies that to aim of studding ordinary differential equations with uncertain determined dynamics it is natural to use differential inclusions as the generalization and a good tool for analyses of properties and behaviours' of systems described by ODE.

The goal of this study is to extend to the fractional order case the Lyapunov based approach to collision avoidance. Since we consider the system starting from given initial state and the kind of history of the system is not consider, the Caputo fractional order derivative is taken into account, [3,4,11].

The paper is organized as follows. In Section 2 the needed notation and facts are presented. Fractional order nonlinear control systems with Caputo differential and its relation with fractional order differential inclusion are introduced. It is shown then the set of trajectories of fractional order continuous-time inclusion associated with the given system is closed. In Section 3 there is considering the avoidance problem. Basing on [14] and [22] conditions for giving the constructive strategy allowing keeping one agent in the avoidance zone no matter what the admissible action of the other agent are given. To this aim the Lyapunov approach is used. As an example of determination the proposed avoidance strategies linear fractional order systems are consider in Section 4.

2. FRACTIONAL CONTROL SYSTEMS AND INCLUSIONS

Let $x\colon [a,b] \to \mathbb{R}$ be an absolutely continuous real valued function. The Caputo fractional derivative of order $\alpha, 0 < \alpha \leq 1$, of a function x is defined by, [11]:

$$ {}_a^C D^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{x\prime(\tau)}{(t-\tau)^\alpha} d\tau, $$

where Γ denotes the gamma function.

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Consider the fractional order differential control system

$$\left({}_a^C D^{\alpha} x(t)\right)(t) = f\big(t, x(t), u(t)\big), \;\; x(0) = x_0 \qquad (1)$$

where $t \in \mathbb{R}_+$, $x(t) = \big(x_1(t) \ldots \; x_n(t)\big)^T \in \mathbb{R}^n$ is a state vector with bounded entries, $\mathrm{u}(\cdot) \in \mathcal{U} \subseteq \mathbb{R}^{\mathrm{m}}$ is a control function defined on the set of admissible controls

$$\mathcal{U} = \{u(\cdot) \text{ mesurable}, u(t) \in U \text{ for all } t \in \mathbb{R}_+\}$$

such that $U \subset \mathbb{R}^m$ is a compact set of control values, $f: \Omega \supseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. We assume that Ω is an open subset of $\mathbb{R} \times \mathbb{R}^n$, the dynamics of system (1), i.e. function $f: \Omega \times U \to \mathbb{R}^n$, of C^1-class with the respect to x and continuous with respect to each other variable.

An absolutely continuous function $x(\cdot): [t_0, t_1] \to \mathbb{R}^n$ is a solution of (1) if its graph $\{(t, x(t)): t \in [t_0, t_1]\}$ is entirely contained in Ω and there exists a measurable u with values inside U, so that $\left({}_{t_0}^C D^{\alpha} x(t)\right)(t) = f\big(t, x(t), u(t)\big)$ for almost every $t \in [t_0, t_1]$. This solution forms the trajectory of system (1). The motion of this system can be described by the multifunction

$$F(x) = \{f\big(t, x(t), u(t)\big): u(\cdot) \in U\}.$$

Consider the fractional order differential inclusion

$$\left({}_0^C D^{\alpha} x\right)(t) \in F(t, x) \qquad (2)$$

Theorem 1: A function $x: [t_0, t_1] \to \mathbb{R}^n$ is a trajectory of (1) if and only if it satisfies (2) almost everywhere.
Proof: In fact the reasoning is similar to the classical continuous-time case, see [5], so we only sketch the proof. The fact that a solution of (1) is the solution of (2) is immediate. Suppose that $x(\cdot)$ is a solution to (2). For the fixed arbitrary element $\bar{\omega} \in U$ let us define the multifunction

$$W(t) = \begin{cases} \{\omega \in U: f(t, x(t), \omega) = \left({}_0^C D^{\alpha} x\right)(t)\} \text{ if} \\ \qquad \left({}_0^C D^{\alpha} x\right)(t)\,(t) \in F(t, x); \\ \{\bar{\omega}\} \qquad \text{otherwise.} \end{cases}$$

The equality $W(t) = \{\bar{\omega}\}$ holds only on a set of the measure zero, so $\left({}_0^C D^{\alpha} x(t)\right)(t) = f(t, x(t), \omega)$ is fulfilled for almost every $t \in [t_0, t_1]$. Define a control u in such way that u(t) is the first element of this set with respect to the lexicographical order. Using the same arguments as in [5,22] one can conclude that function $\mathrm{u}(\cdot)$ is measurable. □
Example: Consider the system

$$\left({}_0^C D^{\alpha} x\right)(t) = u(t), \;\; x(0) = 0 \qquad (3)$$

with $u(t) \in U = \{-1; 1\}$. Then $x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u}{(t-\tau)^{1-\alpha}} d\tau = \frac{t^{\alpha} u}{\Gamma(1+\alpha)}$. Let $u_n = 1$ if $sin(nt) \geq 0$ and $u_n = -1$ if $sin(nt) < 0$. Then the sequence of trajectories $\{x_{u_n}(t)\}$ converges uniformly to zero for any t. On the other hand, it is easy to check $x(t) \equiv 0$ is not a solution of (3).

Below we answer on the question when the limit of system's trajectories coincides with the set of its solutions.

2.1 Closure of the Set of Trajectories

Let $A \subset \mathbb{R}^n$ be a nonempty set. Recall that a distance of point $x \in \mathbb{R}^n$ from set A, denoted by $d(x, A)$, is defined as $d(x, A) := \inf_{a \in A} d(x, a)$. Recall also that a multifunction F with compact values is Hausdorff continuous if $lim_{x_2 \to x_1} d\_H(F(x_2), F(x_1)) = 0$, for every $x_1 \in \mathbb{R}^n$, where d_H is the Hausdorff distance between two nonempty compact sets $F(x_2)$ and $F(x_1)$ from $\mathbb{R}^n$ defined as

$$d_H\big(F(x_2), F(x_1)\big) := max\{d(x, F(x_1), d\big(x', F(x_2)\big): x \in F(x_1),\; x' \in F(x_2$$

Theorem 2: Assume that $(t, x) \mapsto F(t, x)$ is Hausdorff continuous map on $\mathbb{R} \times \mathbb{R}^n$ with compact convex values. Then the set of trajectories of inclusion (2) is closed in $C^0([0, T], \mathbb{R})$.
Proof: Although the basic idea of proof itself comes from [5] there will be needed some facts on approximation of solution to continuous-time fractional order dynamic system by the solution to discrete fractional order dynamic systems, that can be found in the Appendix.

Suppose that $x_n(\cdot)$, $n \in \mathbb{N}$, is a sequence of trajectories of (2) converges uniformly to $x(\cdot)$ on $[0, T]$. Note also that the sets $F(t, x_n)$ are uniformly bounded. This implies that $x_n(\cdot)$ are uniformly Lipschitz continuous. It follows that also function $x(\cdot)$ is uniformly Lipschitz continuous and therefore, absolutely continuous on $[0, T]$. Thus, $x(\cdot)$ is differentiable a.e. on $[0, T]$ in the Caputo sense, see [11].

Suppose that $\left({}_0^C D^{\alpha} x\right)(t)$ exists, but it does not fulfil inclusion $\left({}_0^C D^{\alpha} x\right)(t) \in F(t, x)$. By Separation Theorem there exist $\varepsilon > 0$ and a vector $p \in \mathbb{R}$ such that

$$p\left({}_0^C D^{\alpha} x\right)(t) \geq max_{y \in F(t, x(t))}(py) + \varepsilon \geq py + \varepsilon$$

for all $y \in F\big(t, x(t)\big)$. Let $h > 0$. By continuity there exists $\delta > 0$ such that for $|t - t_h| \leq \delta$ and $|x(t) - x(t_h)| \leq \varepsilon$, one has

$$py \leq p\left({}_0^C D^{\alpha} x\right)(t) - \varepsilon \qquad (4)$$

for all $y \in F(t, x(t))$. By Proposition 6 in Appendix there exists ϵ, for example $\epsilon = \varepsilon$, such that

$$|\left({}_0^C D^{\alpha} x\right)(t) - \left({}_a^C \Delta_h^{\alpha} x\right)(t_h)| < \varepsilon \qquad (5)$$

for h small enough, where ${}_a^C \Delta_h^{\alpha} x$ denotes the Caputo-type difference operator (see Appendix). Then, by the uniform convergence and (5) one gets

$$\lim_{\nu \to \infty} p\left({}_a^C \Delta_h^{\alpha} x\right)(t_h) = p\left({}_a^C \Delta_h^{\alpha} x\right)(t_h) >$$

$$p\left({}_0^C D^{\alpha} x\right)(t) - \varepsilon.$$

Putting $\mathrm{y} = \left({}_a^C \Delta_h^{\alpha} x\right)(t_h)$ from (4) one gets

$$p\left({}_a^C \Delta_h^{\alpha} x\right)(t_h) \leq \left({}_0^C D^{\alpha} x\right)(t) - \varepsilon$$

what leads to the contradiction. □
Corollary 3: Assume that $(t, x) \mapsto F(t, x)$ is Hausdorff continuous map on $\mathbb{R} \times \mathbb{R}^n$ with compact convex values. Let $x_n(\cdot)$ be a sequence of trajectories of (2) converges uniformly to $x(\cdot)$ on $[0, T]$.

i. If the graph $\{(t, x(t)): t \in [0, T]\}$ is entirely contained in Ω and all sets $F(t, x) = \{f(t, x, u): u \in U\}$ are convex, then $x(\cdot)$ is also a trajectory of the system

$$\left({}_0^C D^{\alpha} x\right)(t) = f\big(x(t), u(t)\big), \;\; u \in U\}. \qquad (6)$$

ii. The set of trajectories of inclusion (6) is closed in $C^0([0, T], \mathbb{R})$.

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Proof: Item $i.$) is driven directly from Theorem 2. Item $ii.$) is the consequence of the item $i.$) and Theorem 1. □

3. AVOIDANCE STRATEGY

Let $p_i: \mathbb{R}_+ \cup \{0\} \times \mathbb{R}^n \to U_i \subseteq \mathbb{R}^{d_i}$ $i = 1,2$ be strategies from the given class of set valued functions $\mathcal{U}_i$ with control values u_i ranging in given sets U_i. These strategies could arise from the application of a proper feedback low to the system (1). This implies that $U_i = U_i(t, x(t))$. should be admissible for the given system. So, strategies $p_i, i = 1,2$, have to be such that for given $(t, x(t))$ it holds that $u_i \in p_i(t, x(t)) \subseteq U_i \subseteq \mathbb{R}^{d_i}$ for $i = 1,2$.

Consider a multivalued function

$$F(t,x) := \{\bar{x} \in \mathbb{R}^n : \bar{x} = f(t, x, u_1, u_2), u_i \in p_i(t, x(t)), \quad i = 1,2\}$$

where $f: \mathbb{R}_+ \cup \{0\} \times \mathbb{R}^n \times \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}^n$ is the dynamic of the system (1). This means that

$$({}_0^C D^\alpha x)(t) \in F(t, x(t), p_1(t, x(t)), p_2(t, x(t))) \tag{7}$$

From Theorem 1 it follows that the trajectory of the system (7) is absolute continuous function $x: [t_0, t_1] \to \mathbb{R}^n$ satisfying fractional order inclusion

$$({}_0^C D^\alpha x)(t) \in F(t, x(t), p_1(t, x(t)), p_2(t, x(t)))$$

almost everywhere for $t \in [t_0, t_1]$.

Let Λ be an open or the closure of an open set in $\mathbb{R}^n$. Then for the system (7):

1. a set $\mathcal{T}$ such that $\mathcal{T} \subseteq \Lambda$ and into $\mathcal{T}$ there is no solutions of system (7 that must enter for some $p_1(t, x(t)) \in \mathcal{U}$ no matter what $p_2(\cdot, x(\cdot)) \in \mathcal{U}$ is called the *antitarget set*,
2. a closed set $\mathcal{A} \subseteq \Lambda$ such that $\mathcal{T} \subset \mathcal{A}$ is called the *avoidance set*,
3. a set $\Upsilon_{\mathcal{A}} := \Lambda_\varepsilon \setminus \mathcal{A}$, where Λ_ε is a closure of the open set Λ, such that $\mathcal{A} \subset \Lambda_\varepsilon$ is called the *safety zone*.

Note that $\mathcal{T} \subset \mathcal{A} \subset \Lambda_\varepsilon$ and all these sets are subsets of trajectories set of the considered system. Moreover, the avoidance set can be any set that contains the antitarget set $\mathcal{T}$.

Following [14], let us introduce the following notation. Let $\Phi_{\mathcal{A}} := \mathbb{R}_+ \times \Upsilon_{\mathcal{A}}$. By an *attainable set of trajectories* $\gamma(t, x_0)$ at time $t \geq 0$ we will mean the set

$$\gamma(t, x_0) := \{x(t): \text{given } p_1(\cdot, x(\cdot)) \in \mathcal{U}_1 \text{for all}$$

$$p_2(\cdot, x(\cdot)) \in \mathcal{U}_2 \text{ and given } x_0 \in \Phi_{\mathcal{A}}\}.$$

The set $\gamma(t, x_0)$ is the set of all motions of the system (7) from the initial state $x_0 = x(0) \in \Phi_{\mathcal{A}}$ at time t, i.e. it is the set of all motions of the system (7) from x_0 laying in the safety zone $\Phi_{\mathcal{A}}$. The *fuel motion* $\Gamma(t, x_0)$ from $x_0 \in \Phi_{\mathcal{A}}$ is the set
$\Gamma(t, x_0) := \cup_{t \in \mathbb{R}_+} \gamma(t, x_0)$
and $\Gamma(\Phi_{\mathcal{A}}) := \cup_{x_0 \in \Phi_{\mathcal{A}}} \gamma(\mathbb{R}_+, x_0)$.

Theorem 4: A set $\mathcal{A}$ is the avoidance set for the nonlinear fractional order system (7) if there exist:

1. a set $\Phi_{\mathcal{A}}$,
2. strategy $p_1(\cdot, x(\cdot)) \in \mathcal{U}_1$,
3. continuous real function V defined on an open subset of $\overline{\Phi_{\mathcal{A}}}$ such that for all $(t, x(t)) \in \Phi_{\mathcal{A}}$ it holds:
   1. $V(t, x(t)) > (t, x(t)) V(t_1, x(t_1))$ for $x_1 \in \partial\mathcal{A}$ and $t_1 \geq t$;
   2. $({}_0^C D^\alpha V)(t, x(t)) \geq 0$ for $u_2(t) \in \mathcal{U}_2$ and $\tilde{p}_1((\cdot), x(\cdot)) = p_1(\cdot, x(\cdot))|_{\Phi_{\mathcal{A}}}$.

Proof: Suppose that for some $x_0 = x(0) \in \Phi_{\mathcal{A}}$ there exists $t_2 > 0$ such that $\Gamma(t, x_0) \cap \mathcal{A} \neq \emptyset$ for $t \in [0; t_2]$. By item $i.$) there exist a $t_1 \in (0; t_2]$ and $x_1 = x(t_1) \in \Gamma(t, x_0) \cap \partial\mathcal{A}$ being the end point of the trajectory that lies on the boundary $\partial\mathcal{A}$ such that $V(0, x_0) > V(t_1, x_1)$. Suppose that $u = (u_1, u_2)$ is an admissible control for the system

$$({}_0^C D^\alpha x)(t) \in F(t, x(t)) = \{\bar{x} \in \mathbb{R}^n : \bar{x} = f(t, x, u_1, u_2)\}, \quad x(0) = x_0 . \tag{8}$$

Then $\gamma(t, x_0, u)$ is the solution of (8) describing the trajectory of this system. Consider the maximal solution of

$$({}_0^C D^\alpha x)(t) = f_u(t, x(t)), \quad x(0) = x_0. \tag{9}$$

and suppose that there is a time interval such that (9) has exactly one maximal solution. From the item $i.$) it follows that $V(t, \gamma(t, x_0, u))$ is nondecreasing continuous function as long as $\gamma(t, x_0, u)$ stays in $\Phi_{\mathcal{A}}$. For this it is sufficient that for some ε it remains in the set $\Psi := \{(t, x): V(t, x) \geq \varepsilon\}$.

Suppose that there is some $s > 0$ such that $V(s, \gamma(s, x_0, u)) \leq \varepsilon$. From continuity of V it follows there is a first such s. So, one can suppose that $V(t, x(t)) > \varepsilon$ for all $t \in [0; s)$. Then $t \mapsto V(t, x(t))$ is nondecreasing map on $[0; s]$ and hence $V(s, \gamma(\iota\, x_0, u)) \geq V(t, x(t)) > \varepsilon$, so contradiction. This means that γ stays in the compact set Ψ for all $t > 0$. So, the trajectory is defined for all $t > 0$ and V is nondecreasing. □

4. A LINEAR FRACTIONAL ORDER SYSTEM

As an example let consider a linear continuous-time fractional system with order $0 < \alpha < 1$ and with the Caputo derivative

$$({}_0^C D^\alpha x)(t) = Ax(t) + B_1 u_1(t) + B_2 u_2(t) \tag{10}$$

where state $x(t)$ belongs to some set $\Lambda \subseteq \mathbb{R}^n$, $u_i(t) \in \mathcal{U}_i \subseteq \mathbb{R}^{d_i}$, $i = 1, 2$, $t \in [t_0 = 0, t_1]$ and A, B_1, B_2 are stationary matrices of appropriated dimensions.

Recall that system $({}_0^C D^\alpha x)(t) = Ax(t)$ is [7]

1. *asymptotically stable* iff $|arg(specA)| > \frac{\pi}{2}\alpha$,
2. *stable* iff either it is asymptotically stable or those critical eigenvalues which satisfy $|arg(specA)| = \frac{\pi}{2}\alpha$ have geometric multiplicity one.

In [2] it was shown that linear system
$({}_0^C D^\alpha x)(t) = Ax(t) + Bu(t)$
is controllable on $t \in [0, t_1]$ iff adjoint linear system
$({}_{t_1}^C D^\alpha x)(t) = A^T x(t),$
$y(t) = B^T x(t)$
is observable on this interval. Then the matrix (observability gramian)

$$W(t) = \int_0^t E_\alpha(A(t_1 - \tau)^\alpha) BB^T E_\alpha(A^T(t_1 - \tau)^\alpha) d\tau \tag{11}$$

where $E_\alpha(A\tau^\alpha) = \sum_{k=0}^{\infty} \frac{(A\tau^\alpha)^k}{\Gamma(k\alpha + 1)}$ denotes the Mittag-Leffler matrix function, is symmetric positive define and there exists the positive define symmetric matrix such that $AW + WA^T \geq -Q$.

One can choose the matrix function $V(t) = x^T(t)Px(t)$ with $P = W$. Taking the avoidance set $\mathcal{A} = \{x \in \mathbb{R}^n : x^T Px \leq \mathrm{const}\}$ the condition $ii.)$ of Theorem 4 is fulfilled. For such avoidance set, the antitarget set $\mathcal{T}$ should belong to a ball containing the set $\{0\}$. Moreover, taking a stationary matrix D such that $B_2 = B_1 D$, and defining set $U_i, i = 1, 2$, as

$$U_i := \{u_i : ||u_i|| \leq \xi_i, \xi_i > 0\}$$

with $\xi_1 \geq \xi_2 \|D\|$, where $||\cdot||$ denotes the Euclidian norm, one meets the condition $ii.)$ of Theorem 4. Then, the avoidance strategy can be designed as

$$p_1(t,x) = \frac{B_1^T Px}{\|B_1^T Px\|} \xi_1$$

for all $(t,x) \notin \Sigma = \mathbb{R}_+ \times \{x \in \mathbb{R}^n : DE_\alpha(A\tau^\alpha) = 0 \text{ for all } t\}$. If $(t,x) \in \Sigma$, then u_1 can be admissible control, so $p_1(t,x) = U_1$.

Example: Let us consider the following system

$$({}_0^C D^{0,5} x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_2(t) \qquad (13)$$

with $U_2 = \{u_2 : |u_2| \leq 1\}$ and $t \in [0;1]$. Following [2] it can be calculated that at $t = 0$ the Mittag-Lefflet function is as follows

$$E_{0,5}(At^{0,5}) = \begin{bmatrix} \frac{E_{0,5}(t^{0,5}) + E_{0,75}(-t^{0,5})}{2} & \frac{E_{0,5}(t^{0,5}) + E_{0,5}(t^{0,5})}{2} \\ \frac{E_{0,5}(t^{0,5}) + E_{0,5}(t^{0,5})}{2} & \frac{E_{0,5}(t^{0,5}) + E_{0,5}(t^{0,5})}{2} \end{bmatrix}$$

and the system

$$({}_0^C D^{0,5} x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t)$$

is not stable, but system (13) is controllable, so the respective adjoint linear system is observable. The

$$W = \int_0^t E_{0,5}(A(1-\tau)^{0,5}) BB^T E_{0,5}(A(1-\tau)^{0,5}) \quad d\tau = \begin{bmatrix} 1{,}7036 & 2{,}2882 \\ 2{,}2882 & 3{,}1945 \end{bmatrix}$$

The stabilizing feedback matrix gain is, see [2]

$$K = B^T W^{-1} E_{0,5}(A^T {t_1}^{0,5}) = [-11{,}2277 \quad -2{,}9587]$$

Provided the avoidance set $\mathcal{A} = \{x \in \mathbb{R}^2 : x^T Px \leq a, a \in \mathbb{R}_+\}$ the avoidance strategy is $p_1(t,x) = Kx(t) + \frac{B_1^T Wx}{\|B_1^T Wx\|} \|D\| \xi_1$, where D is such that $B_2 = B_1 D$. Taking D as identity matrix, we obtain

$$p_1(t,x) = -10{,}6694 x_1 - 2{,}1793 x_2.$$

5. CONCLUSIONS

In the paper collision avoidance control strategy for fractional-order dynamic systems with Caputo derivative is developed. Based on Lyapunov stability method the constructive avoidance conditions of determining strategy $p_1(\cdot, x(\cdot)) \in \mathcal{U}_1$ of one agent from the given initial state $x_0 = x(0)$ such that its collision-free trajectories intersects the avoidance set no matter what strategy of the second agent's is, are given. As example how the obtained avoidance strategies can be easily used it is explained on the linear fractional order system. As an intermediate step it is shown that the set of trajectories of given system are closed in the set of continuous functions on finite time interval. Our future research directions include extending these results to non-cooperative multi-agents fractional case and next for fractional-order dynamic systems with Riemann-Liouville and Grünwald--Letnikov fractional-order operators.

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APPENDIX 1

Fractional order discrete-time systems

It is known that if $x: h\mathbb{Z} \to \mathbb{R}$ then forward h −difference operator is defined as $(\Delta_h x)(kh) = \frac{x(k+h)-x(k)}{h}$. If $q \in \mathbb{N}_0 := \{0,1,2,\dots\}$, then $\Delta_h^q := \Delta_h \circ \cdots \circ \Delta_h$ denotes the q −fold application of the operator Δ_h, i.e. $(\Delta_h^q x)(kh) =$
$= \sum_{i=0}^{q}(-1)^{q-1}\binom{q}{k} x(kh+ih)h^q$. The extension of q-fold application of operator Δ_h leads to the fractional h -sum:

$$(\Delta^{-\alpha_h} x)(kh) := \sum_{i=0}^{k} h^\alpha a^{(\alpha)}(i) x(kh-ih) \qquad (1)$$

where $k \in \mathbb{N}_0$ and $a^{(\alpha)}(i) = (-1)^i \frac{\alpha(\alpha-1)\cdots(\alpha-i+1)}{i!}$.

Definition 1: Let $\alpha \in (q-1,q]$, $q \in \mathbb{N}$. Then, the *Caputo - type fractional* $h -$ *difference* of order α for a function $x: h\mathbb{Z} \to \mathbb{R}$ is defined as

$$(\Delta^{\alpha_h} x)(kh) = \left(\Delta_h^{-q-\alpha}\left(\Delta_h^q x\right)\right)(kh) \qquad (2)$$

From Definition 5 it follows that

1. if $q = 1$ then $(\Delta^{\alpha_h} x)(kh) = (\Delta^{-(1-\alpha)_h}(\Delta_h x))(kh)$,
2. if $\alpha = q \in \mathbb{N}$, then $(\Delta^{\alpha_h} x)(k) = \left(\Delta_h^q x\right)(k) = h^q \sum_{i=0}^{q}\binom{q}{k} x(k+i)$.

Consider the discrete fractional system of the form

$$(\Delta_h^\alpha y)(kh) = f\big(kh, y(kh), u(kh)\big),\ y(0) = y_0 \qquad (3)$$

where $y(\cdot) \in \mathbb{R}^n, k = \lfloor\frac{t}{h}\rfloor + 1$ with the sign $\lfloor\cdot\rfloor$ being the floor function and $u(\cdot)$ - a control vector function defined on the set $\mathcal{V} = \{u(\cdot)\ \text{measurable}, u(kh) \in V \text{ for all } k \in \mathbb{N}\} \subseteq \mathbb{R}^m$ with the set of control values $V \subset \mathbb{R}^m$ being compact. We say that a function $y(\cdot)$ defined on a set $\{\kappa h, (\kappa+1)h, \dots, kh\}$, $\kappa < k$, $\kappa \in \mathbb{N}$, is a solution of (16) if its graph $\{(nh, y(nh)): \kappa h \le nh \le kh\}$ is entirely contained in $\Omega_h \subseteq (h\mathbb{N})_a \times \mathbb{R}^n$, and there exists a measurable control function $u(\cdot)$ with values inside V, so that $({}_a\Delta^{\alpha_h} y)(kh) = f(kh, y(kh), u(kh))$ for almost every $t \in \{\kappa h, (\kappa+1)h, \dots, kh\}$. This solution forms the k −steps trajectory of system (16).

Proposition 1: Suppose that state function $x: \mathbb{R} \to \mathbb{R}^n$ is absolutely continuous. Then the solution x to system

$$({}_0^C D^\alpha x)(t) = f\big(t, x(t)\big),\ x(0) = x_0, t \in (0,T]$$

can be approximated by the solution of system:

$$(\Delta_h^\alpha y)(kh) = f(kh, y(kh)), y(0) = y_0, t \in (0,T]_{(h\mathbb{N})_a}$$

where $k = \left\lfloor\frac{t}{h}\right\rfloor + 1$ with in values via the limit $\lim_{h\to 0} y(kh) = x(t)$.